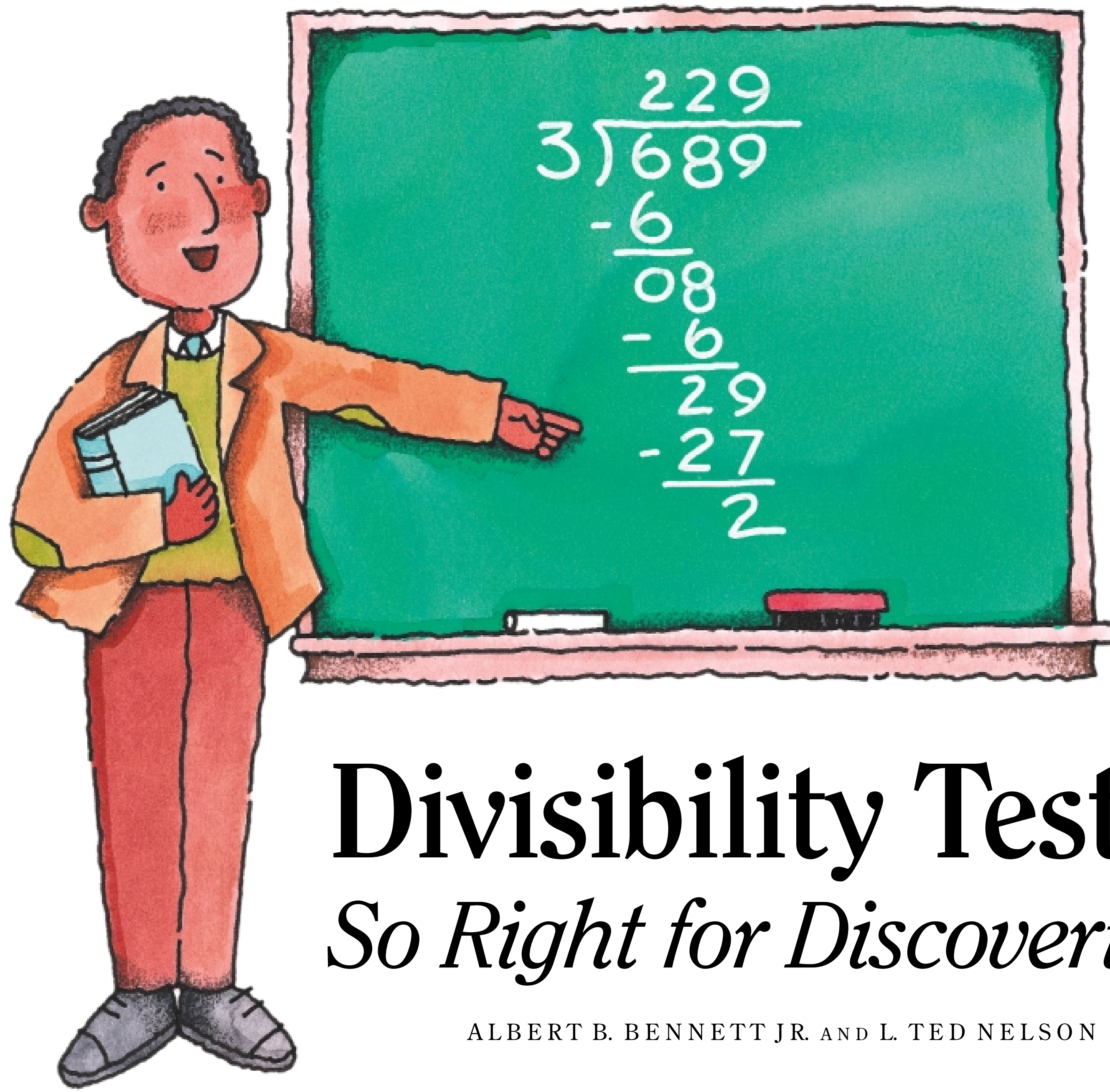

Divisibility Tests:
So Right for Discoveries

ALBERT B. BENNETT JR. AND L. TED NELSON

OVER THE YEARS, TEACHERS HAVE WRITten to journals to share mathematical discoveries made by their students. Often these involve number patterns or relationships that seem to be true but that students can justify only with numerical examples. One such example, related by seventh-grade teacher Robert Ruble in "Readers Write" (Ruble 1999), describes a rule that was discovered by Sarah Martin, one of his students: "When you double the tens digit of a two-digit number and add the ones digit, if the sum is divisible by 8, then so also is the original number. For a three-digit number, take the hundreds digit with the tens digit and double them and add the ones digit." For example, 96 is divisible by 8 because $2 \times 9 + 6 = 24$, and 24 is divisible by 8; 176 is divisible by 8 because $2 \times 17 + 6 = 40$, and 40 is divisible by 8. This article illustrates how base-ten pieces can be used to make divisibility rules clearer to students and to promote discovery of traditional divisibility tests and students' own divisibility tests.

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Determining Divisibility with Base-Ten Pieces

FIGURE 1 HELPS US EXAMINE "SARAH'S RULE" for determining if 96 is divisible by 8 by showing the base-ten pieces for 96, 9 longs and 6 units. Use the measurement concept of division ("take away" concept) to remove as many groups of 8 units as possible from each long. This action leaves 2 units in each long; the number of units remaining is $2 \times 9 + 6 = 24$, which is divisible by 8.

To illustrate why Sarah's test works for integers with more than two digits, consider the base-ten representation for 136 (see **fig. 2**), namely, 1 flat, 3 longs, and 6 units. Because each flat (10×10 base piece) is equivalent to 10 longs, these pieces can be regrouped with the other pieces to give 13 longs and 6 units. Removing 8 units from each of the 13 longs leaves 2 units in each. The number of units remaining is $2 \times 13 + 6 = 32$, which is divisible by 8, as is 136.

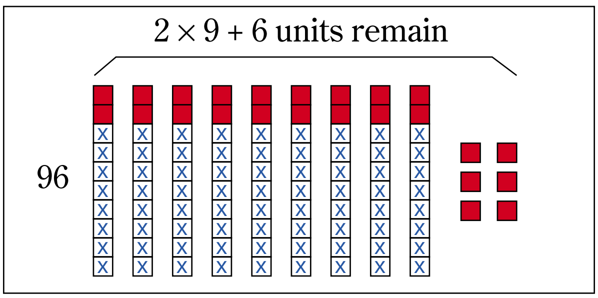

Fig. 1 Remove 8 units from each of the 9 longs, leaving $2 \times 9 + 6 = 24$ units. Because 24 is divisible by 8, we know that 96 is also divisible by 8.

One of Sarah's classmates discovered that Sarah's test also works for divisibility by 4. The diagrams in **figures 1** and **2** illustrate divisibility by 4 because removing a group of 8 is the same as removing two groups of 4.

By representing numbers with base-ten pieces, students can find a variety of ways to answer divisibility questions. **Figure 3** shows the base-ten representation for 376 and illustrates a second method for determining divisibility by 8. Suppose the following question is posed: "If 376 units are given away in blocks of 8 units at a time, will any units be left over?" Marking off groups of 8 units from each flat and long may lead students to observe that when removing groups of 8 units from each flat, 4 units remain, and when removing 8 units from each long, 2 units remain. Therefore, to determine if a number is divisible by 8, multiply the hundreds digit by 4, add 2 times the tens digit, then add the units digit. If this sum is divisible by 8, so is the original number. This test also works for numbers with more than three digits because 1000 and higher powers of 10 are all divisible by 8.

Considering the divisibility of powers of ten, as in **figure 3,** leads to other divisibility tests. For example, one test for divisibility by 6 is similar to the test for divisibility by 8 in **figure 3.** The method of looking at

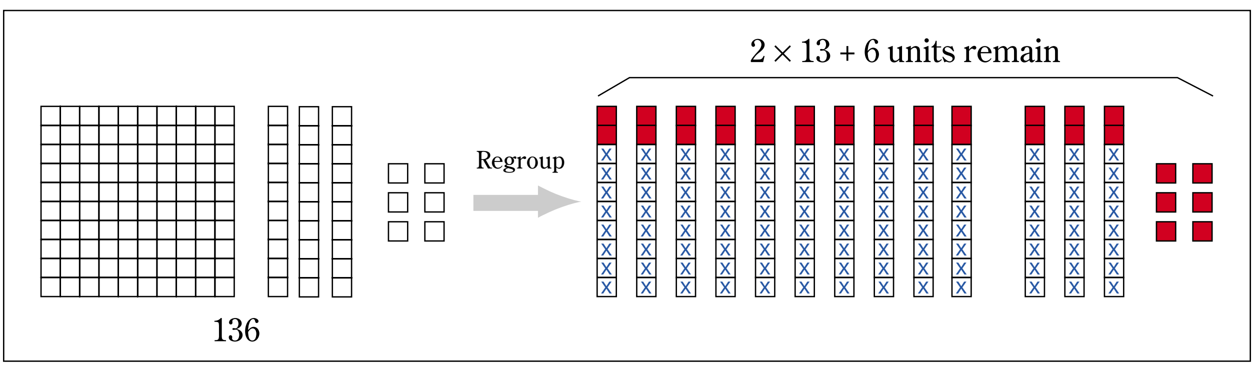

Fig. 2 Remove 8 units from each of 13 longs, leaving $2 \times 13 + 6 = 32$ units. Because 32 is divisible by 8, we know that 136 is divisible by 8.

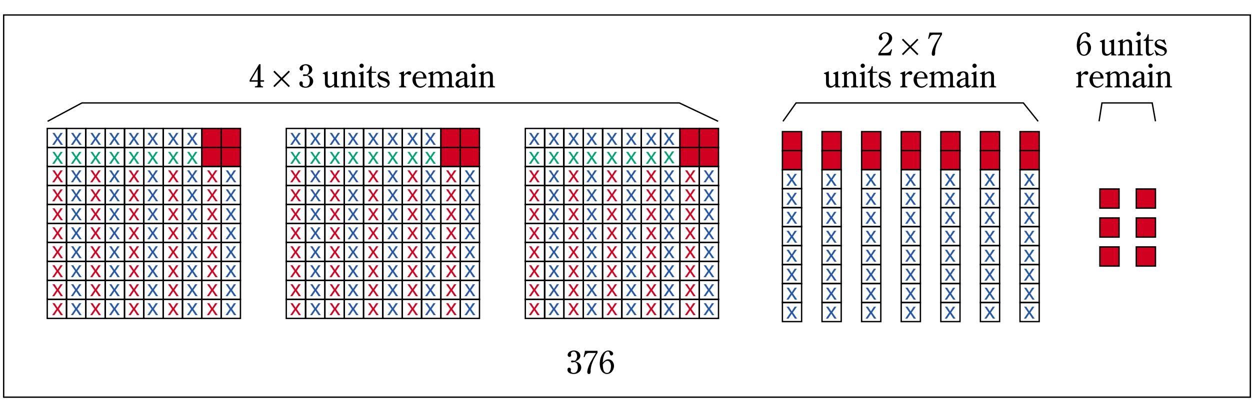

Fig. 3 Because $4(3) + 2(7) + 6 = 32$ and 32 is divisible by 8, the original number, 376, is divisible by 8.

powers of ten is also used in **figure 4** to determine if 478 is divisible by 9. This example illustrates the well-known rule that a number is divisible by 9 if the sum of its digits is divisible by 9. A similar rule exists for divisibility by 3 because each base-ten piece for a power of ten also leaves a remainder of 1 when divided by 3.

An important concept in all divisibility tests is that the original number to be tested is replaced by a new, smaller number. When using base-ten pieces to determine divisibility, students will also be able to see that any remainder obtained from the new, smaller number will be the same as the remainder from the original number. For example, **figure 4** shows that 478 is not divisible by 9, but because groups of 9 units were removed, it also shows that the reminder of 1 from dividing 19 by 9 is the same remainder that will be obtained when dividing 478 by 9.

Although the divisibility tests can be illustrated by using the measurement (take-away) concept of division, in some situations, using the sharing concept of division is easier. For example, because each of the base-ten pieces for the powers of ten can be divided into two equal parts, determining if a number is divisible by 2 is only a matter of checking the units digit. A similar observation can be made for divisibility by 5. **Figure 5** illustrates the sharing concept for the division of 1236 and shows that this number has a remainder of 1 when divided by 5.

Tests for divisibility by 7 are not as well known as divisibility tests for other single-digit divisors; how-

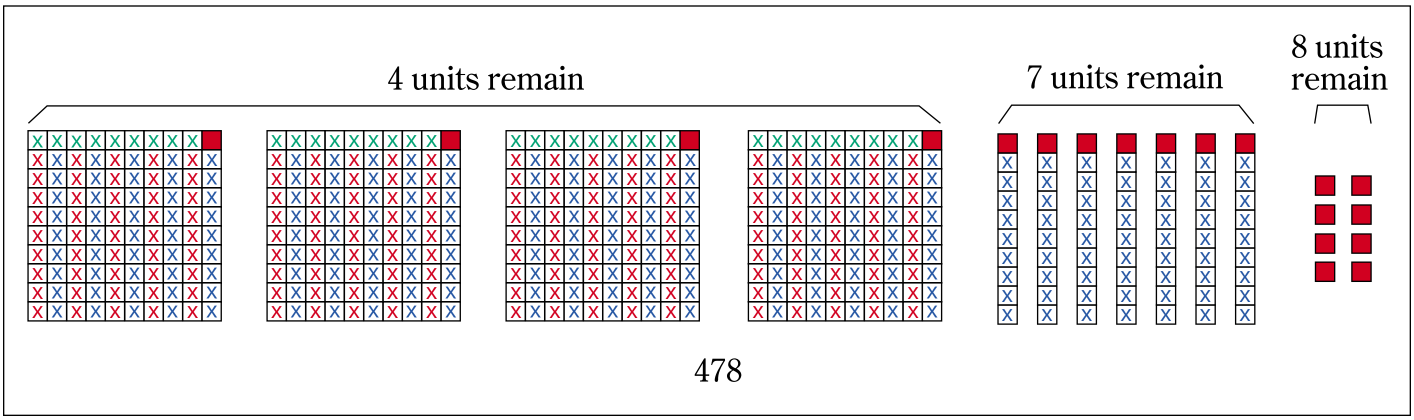

Fig. 4 Each flat and each long leave a remainder of 1 unit when divided by 9; therefore, to determine divisibility by 9, check to see if the sum of the number of flats, longs, and units is divisible by 9. Because 4 + 7 + 8 = 19, which is not divisible by 9, neither is 478.

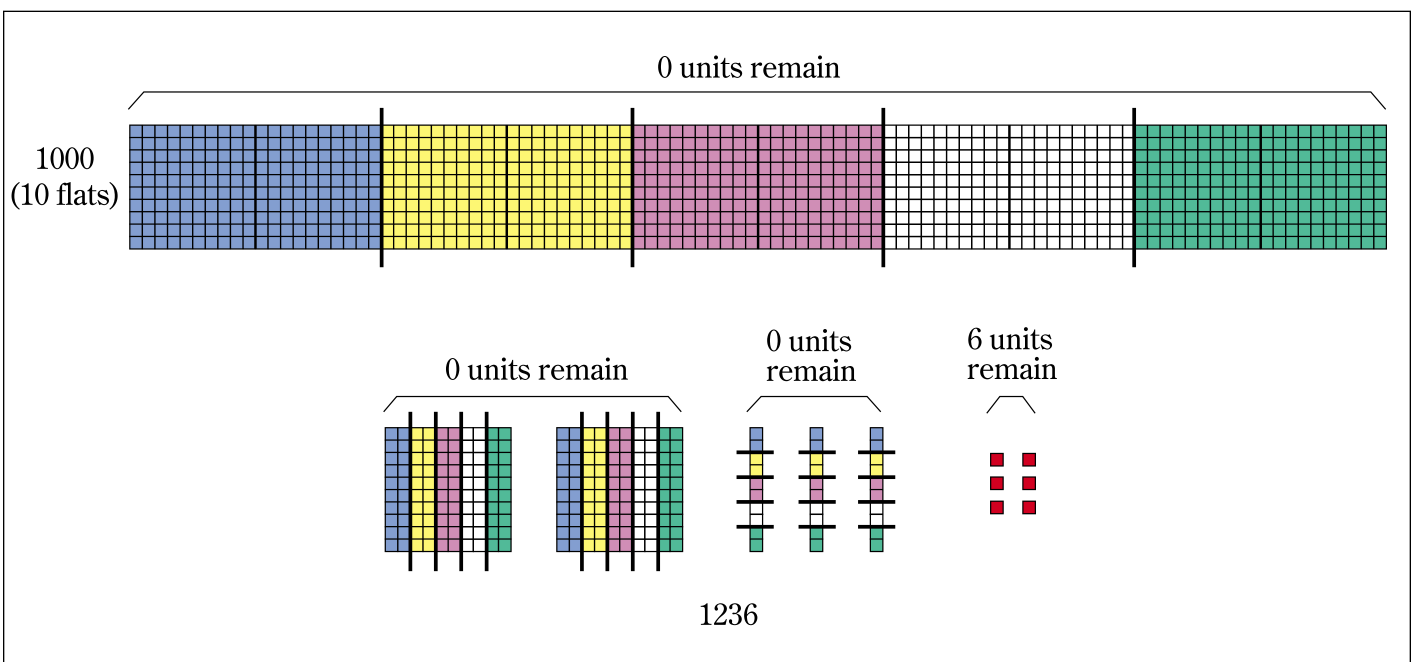

Fig. 5 Each base-ten piece for 10, 100, and 1000 can be divided into five equal parts; therefore, considering only the units digit is necessary to determine if a number is divisible by 5. Because 6 is not divisible by 5, neither is 1236.

ever, suggesting that students try to find easy methods of determining divisibility by 7 with base-ten pieces can lead to surprising results. One possibility is to use an approach similar to Sarah's test. Consider the base-ten representation for 161 (see **fig. 6**). Removing 7 units from each of the 6 longs and the 10 longs in the flat leaves 3 units from each. The number of units remaining, therefore, is 3 × 16 + 1 = 49; because 49 is divisible by 7, so is 161. In other words, to determine if a number is divisible by 7, add the units digit to 3 times the number formed by the remaining digits and test the result for divisibility by 7.

Students might discover a second method for determining divisibility by 7 by noticing that dividing a

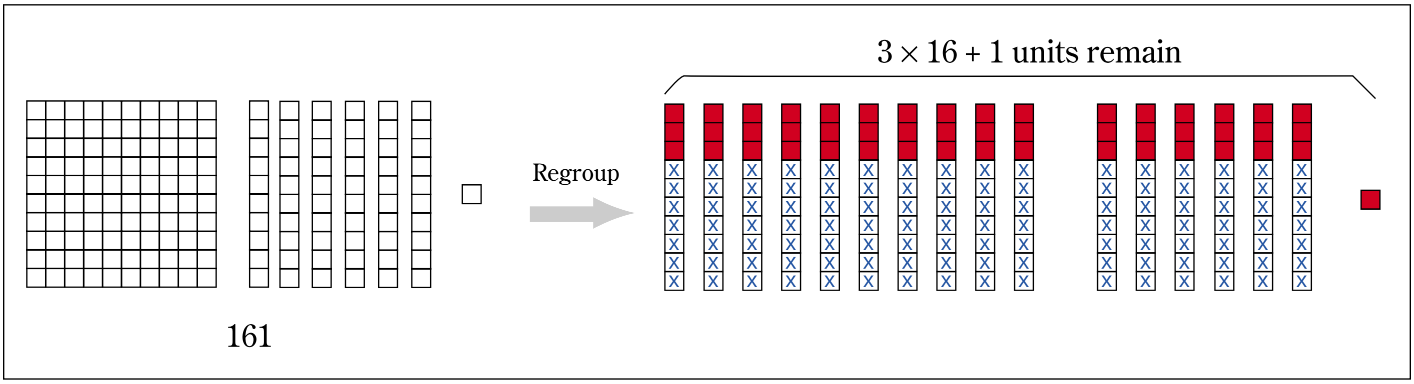

Fig. 6 Remove 7 units from each of the 6 longs and the 10 longs in the flat; because the number of remaining units, 3 × 16 + 1 = 49, is divisible by 7, so is the original number, 161.

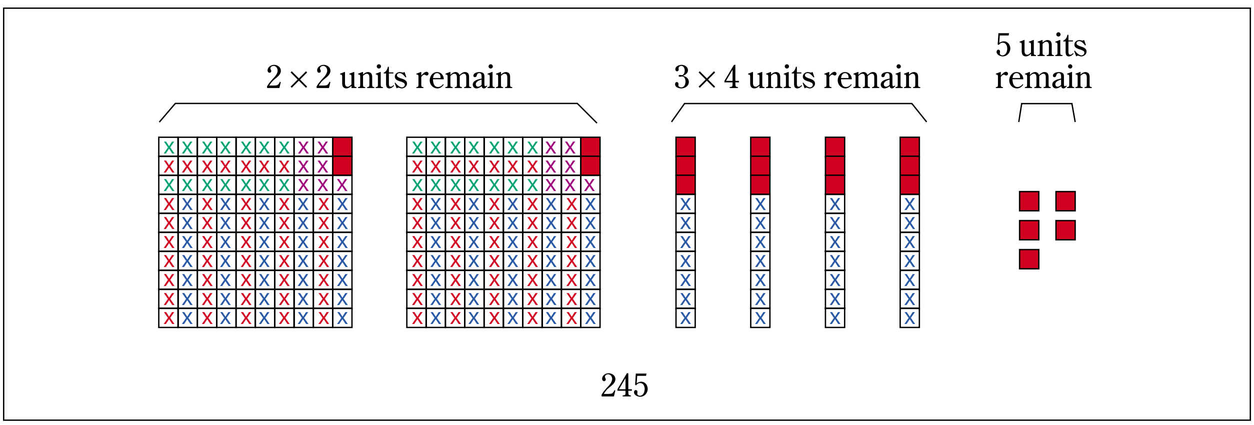

Fig. 7 Each flat leaves a remainder of 2 units when divided by 7, and each long leaves a remainder of 3 units. Because the number of remaining units, 2(2) + 3(4) + 5 = 21, is divisible by 7, so is the original number, 245.

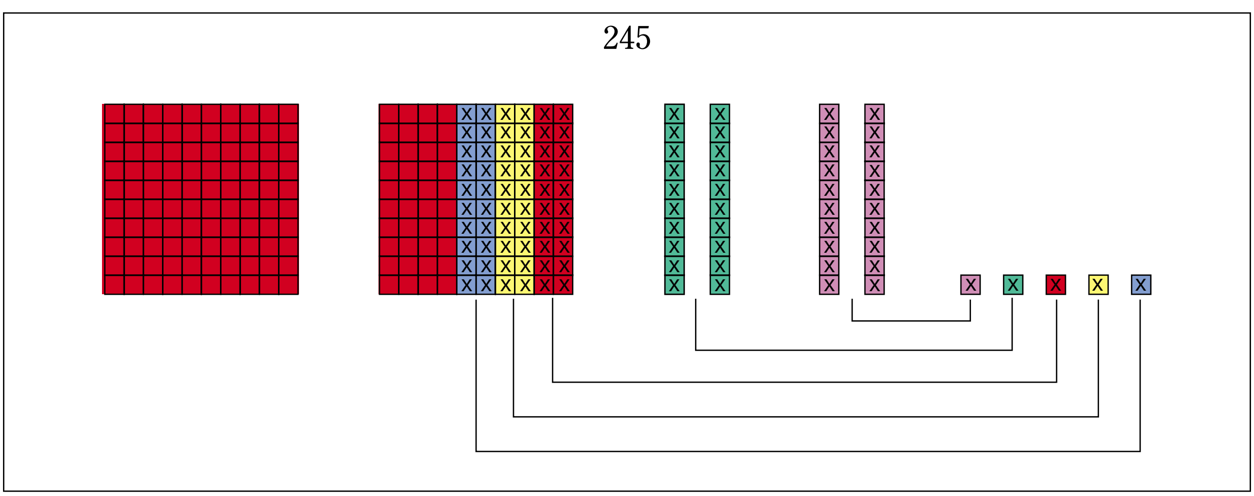

Fig. 8 Groups of 2 longs and 1 unit are removed five times. Because the number represented by the remaining base-ten pieces is divisible by 7, 245 is also divisible by 7.

flat by 7 leaves a remainder of 2 units and dividing a long by 7 leaves a remainder of 3 units (see **fig. 7**). To determine if a three-digit number is divisible by 7, add 2 times the hundreds digit and 3 times the tens digit to the units digit and determine if this sum is divisible by 7. For example, 245 is divisible by 7 because $2(2) + 3(4) + 5 = 21$, which is divisible by 7.

In the preceding two tests for divisibility by 7, the measurement concept of division was used to remove groups of 7 units. Some students might notice that these examples have convenient combinations of base-ten pieces. For example, removing groups of 7 units from a collection of 2 longs and 1 unit leaves $3 + 3 + 3 = 7$ units; the number represented by 2 longs and 1 unit, namely 21, is divisible by 7. **Figure 8** shows that groups of 2 longs and 1 unit are removed five times from the base-ten pieces for 245. The remaining flat and 4 longs represent 140, which is divisible by 7.

Ruble notes that one of his seventh-grade students was familiar with the traditional test for divisibility by 7: "To determine if a number is divisible by 7, remove the right hand digit to create a new number, double this digit, and subtract this double from the new number. Then test the resulting number" (Ruble 1999a). To test 245, for example, the result would be $2 \times 5 = 10$ and $24 - 10 = 14$. Because 14 is divisible by 7, so is 245.

The traditional test for divisibility by 7 can also be understood from the activity described above that involved removing groups of 2 longs and 1 unit. In **figure 8,** groups of 2 longs and 1 unit are removed five times from 245. Because 10 longs and 5 units represent 105, this process can be written as the following difference:

245	
−105	Subtract the units digit and its double.
140	Because 140 is divisible by 7, so is 245.

Notice that writing the 0 in 140 is unnecessary when testing the new number for divisibility by 7. After the units digit 5 is doubled, the 0 is dropped, as shown in the following algorithm.

24~~5~~	Drop off the units digit and subtract its
−10	double from the remaining number.
14	Because 14 is divisible by 7, so is 245.

When using the traditional test for divisibility by 7, and in all the other divisibility tests, if divisibility of the new, smaller number that is obtained cannot be easily determined, the test can be carried out on the new, smaller number.

Conclusion

IN A SECOND LETTER TO "READERS WRITE" (Ruble 1999b), Ruble notes that his students have always shown an interest in divisibility tests and have asked about such tests for other numbers. He encourages textbook authors and teachers to consider presenting more material on divisibility so that students understand that these tests exist. Activities using base-ten pieces will help students discover their own divisibility tests and gain a clearer understanding of traditional divisibility tests. These activities also help students acquire number sense for place value, regrouping, and concepts of division. Students' methods for determining divisibility can be stated initially in words involving the base-ten pieces, then later in terms of the digits of a number, and, finally, as generalizations with variables.

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Coining Some Mathematics: The 50 State Quarters Program

SINCE 1999, THE U.S. MINT HAS BEEN GIVING POCKET CHANGE A NEW look, with the advent of its 50 State Quarters program. Those new shiny quarters can also be taken into the classroom in the form of lesson plans.

In 1999, the U.S. Mint began a ten-year program of commemorating each of the nation's states in the order in which they ratified the Constitution. Each year from 1999 to 2008, five state coins will be produced in ten-week increments at the Philadelphia and Denver mints, then never to be produced again. In 1999, Delaware led off the new quarter program, followed by Pennsylvania, New Jersey, Georgia, and Connecticut. In 2008, the last five states to be represented, in order, will be Oklahoma, New Mexico, Arizona, Alaska, and Hawaii. The portrait of George Washington continues to appear on the obverse (heads) side of the quarter, and the new state design is displayed on the reverse (tails) side of the quarter. After all fifty quarters have been produced, the standard "eagle" quarter will go back into production. For a state-by-state schedule for new quarter releases, go to www.usmint.gov/mint_programs/50sq_program/index.cfm?action=schedule.

In keeping with the educational initiative of this program, the U.S. Mint has produced a series of lesson plans. Every year until the end of the 50 State Quarters Program, the U.S. Mint will develop three sets of lesson plans, one each for kindergarten and first grade, second and third grade, and fourth through sixth grade using the new quarters for that year. The content for the 50 State Quarters Program Education Initiative Lesson Plans is designed by a panel of elementary school teachers and reviewed by education experts.

Each booklet includes six individual lesson plans, with teachers' pages, reproducible handouts, background information, and answer keys focusing mainly on social studies and mathematics lessons, with ties to other subjects, such as language arts. The lessons draw on the standards from the National Center for History in the Schools, the National Council for Geographic Education, the Center for Civic Education, and the National Council of Teachers of Mathematics.

For more 50 State Quarters Program information, visit the U.S. Mint Web site at www.usmint.gov and access "Mint Programs," then "50 State Quarters Program." The lesson plans can be downloaded directly from www.usmint.gov/mint_programs/index.cfm?action=educational_initiative.